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Onsager's Reaction Field for the Potts Model from the Path Integral

A. L. Kholodenko[1]

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A new path integral formulation for the q-state Potts model is proposed. This formulation reproduces known results for the Ising model ($q=2$) and naturally extends these results for arbitrary q. The mean field results for both the Ising and the Potts models are obtained as a leading saddle point contribution to the corresponding functional integrals, while the systematic computation of corrections to the saddle point contribution produces the Onsager reaction field terms, which for $q=2$ coincide with results already known for the Ising model.

KEY WORDS: Ising model; Potts model; functional integrals.

1. INTRODUCTION

Use of path integral methods for the Ising model is standard by now.[1] Some time ago, Zia and Wallace extended the path integral treatments for the Ising model to the case of the Potts model.[2] Their work was subsequently accepted and widely used by researchers in the field of critical phenomena.[3,4] Although the results of their work are quite suitable for the subsequent field-theoretic renormalization group analysis, the treatment of the Potts model which they have developed considerably differs from the case of the Ising model as far as the noncritical properties of both models are concerned. Indeed, for the case of the Ising model, it is rather straightforward to obtain the familiar mean field results for arbitrary permissible values of the order parameter. These mean field results can be corrected by the Onsager reaction field terms to be described below. With these corrections, the mean field results are widely used in the theory of spin glasses, for example, within the framework of the TAP approach.[6]

[1] 375 Hunter Laboratories, Clemson University, Clemson, South Carolina 29634-1905.

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This is not the case so far for the Potts spin glass,[7] where the analogous extension of the mean field results is lacking, so that one is forced to begin with the free energy functional, which has only a plausible resemblance to what is considered to be the improved mean field approximation to the Potts model.

So far, the Onsager reaction field corrections to the mean field results have been obtained on the basis of intuitive physical arguments which are very transparent as far as the Ising model is concerned, as it described below in Section 2. The same arguments become much more vague when applied to the Potts model. Recently, Dekeyser *et al.*[8] made an attempt to extend Onsager's original idea to the case of the Potts model. They were motivated by the observation that the mean field treatment of the Potts model produces the first-order phase transition for $q>2$[9] for *all* dimensions, while it is well known[10] that for $d=2$ the first-order phase transition takes place for $q>4$ and is continuous for $q=4$. So they hoped that the inclusion of the Onsager reaction terms will *raise* the lower mean field estimate, $q>2$. Unfortunately, they have not succeeded in providing systematic corrections to the mean field results which contribute to the Onsager reaction field.

Here, I provide a systematic derivation of the Onsager reaction field terms for the Potts model in complete analogy with similar results for the Ising model, which I also rederive with the use of the path integrals methods. This work is organized as follows. In Section 2, the path integral method for the Ising model is discussed in details sufficient for the purposes of reproducing the Onsager reaction field corrections. The mean field results for arbitrary permissible values of the order parameter m_i are obtained as a leading saddle point contribution to the functional integral, while the Onsager corrections come from the treatment of fluctuations around the mean field. The ideas and methods of this section are extended for the case of the Potts model in Section 3, where a new path integral presentation of the Potts model is given along with the rederivation of the known mean field results for arbitrary values of m_i. This rederivation was not available with the use of the other known path integral methods for the Potts model.[2] Computation of the fluctuations around the mean field is presented for the case of the Potts model in Section 4. In this section closed-form, one-loop corrections to the mean field results are obtained along with the Onsager reaction field terms. It is demonstrated that the fluctuation corrections at the one-loop level *do not* change the mean field conclusion about the first-order phase transition taking place for $q>2$. Section 5 is devoted to a brief discussion.

2. PATH INTEGRAL REPRESENTATION FOR THE ISING MODEL AND THE ORIGIN OF THE ONSAGER REACTION FIELD

The mean field magnetization m_i at the site i for the case of the Ising model placed on a d-dimensional hypercubic (for simplicity) lattice with the total number of sites N_0 can be obtained as the solution of equation

$$m_i = \tanh(\beta \tilde{h}_i) \tag{2.1}$$

where $\tilde{h}_i = H_i + \sum J_{ij} m_j$, with h_i an external field at the site i, and β an inverse temperature. Typically, the lattice coupling constant J_{ij} is nonzero only for the nearest neighbors. Equation (2.1) provides an exact solution for the Ising model in cases (a) when the range of interactions J_{ij} is infinite or (b) when the dimensionality d (or the coordination number z of the lattice) goes to infinity. If the coupling constants J_{ij} are independent random variables (usually Gaussian distributed), Eq. (2.1) becomes incorrect even for the case of infinite range of interactions (at least for the Sherington–Kirkpatrick type of spin glass,[11] for example) and need to be improved.[12] The need for such improvement can be understood based on the following physical reasons. The magnetization m_i at the site i comes (a) from the magnetizations of the neighbors of m_i and (b) from m_i itself. The last contribution should be excluded by introducing the reaction field as was first pointed out by Onsager.[13] This can be achieved by means of introducing the local susceptibility χ_{ii} at the site i,

$$\chi_{ii} = \beta(1 - m_i^2) \tag{2.2}$$

With such a susceptibility one can write the following result for the field h_i:[12]

$$\tilde{h}_i = h_i + \sum_l J_{il}(m_l - m_i \chi_{ll} J_{il}) \tag{2.3}$$

The field h_i given in Eq. (2.1) needs to be replaced by that given by Eq. (2.3).

Although the above arguments are very plausible for the case of the Ising model, their extension to the case of Potts model is by no means systematic.[8] Here, I demonstrate that development of the path integral formalism for the Ising model provides a systematic way of obtaining the Onsager reaction field terms and is easily generalizable to the case of the Potts model to be discussed below.

To proceed, the following steps are necessary. First, the combined use of Eqs. (2.1) and (2.3) produces the following equation:

$$\tanh^{-1} m_i = \beta \sum_j J_{ij} m_j - \beta^2 \sum_j J_{ij}^2 (1 - m_j^2) m_i + h_i \tag{2.4}$$

Second, Eq. (2.4) can be obtained as a first functional derivative (with respect to m_i) of the free energy functional[5]

$$\mathscr{F}[\{m_i\}] = -\frac{1}{2} \sum_{i,j}' J_{ij} m_i m_j + \frac{1}{\beta} \sum_i \left[\left(\frac{1+m_i}{2} \right) \ln \left(\frac{1+m_i}{2} \right) + \left(\frac{1-m_i}{2} \right) \ln \left(\frac{1-m_i}{2} \right) \right] - \frac{1}{4\beta} \sum_{i,j}' (1-m_i^2) J_{ij}^2 (1-m_j^2) \tag{2.5}$$

Here and below the prime on the summation sign indicates that the diagonal terms are being omitted. Third, Eq. (2.5) can be obtained from the path integral representation of the Ising model. Following ref. 14, we can write the partition function Z for the Ising model as

$$Z = \mathscr{N} \int \prod_{i=1}^{N_0} d\varphi_i \exp\{-\beta S[\varphi_i, h_i]\} \tag{2.6}$$

where $\mathscr{N}$ is some unimportant normalization factor and the action S is given by

$$S[\varphi_i, h_i] = \frac{1}{2} \sum_{i,j} (\varphi_i - h_i) J_{ij}^{-1} (\varphi_j - h_j) - \frac{1}{\beta} \sum_i \ln[2\cosh(\beta\varphi_i)] \tag{2.7}$$

In arriving at the result (2.7), some sort of regularization for the matrix J_{ij} is required, in principle, as is explained in ref. 15. This regularization, however, is not going to affect the results presented below. Application of the saddle point method to Z produces

$$Z \approx \exp\{-\beta S[\bar{\varphi}_i, h_i]\} \tag{2.8}$$

where the mean field $\bar{\varphi}_i$ is obtained as solution of the saddle point equation

$$0 = \left.\frac{\delta S}{\delta \varphi_i}\right|_{\varphi_i = \bar{\varphi}_i} = \sum_j J_{ij}^{-1} (\bar{\varphi}_j - h_j) - \tanh(\beta \bar{\varphi}_i) \tag{2.9}$$

or

$$\bar{\varphi}_i = h_i + \sum_j J_{ij} \tanh(\beta \bar{\varphi}_j) \tag{2.10}$$

Since in this approximation the free energy $F \approx S[\bar{\varphi}_i, h_i]$, one obtains for the mean field magnetization

$$m_i = -\frac{\delta S}{\delta h_i} = \sum_j J_{ij}^{-1}(\bar{\varphi}_j - h_j) \tag{2.11}$$

Combining Eqs. (2.9) and (2.11), one obtains $m_i = \tanh(\beta\bar{\varphi}_i)$ (or $1/\beta \tanh^{-1} m_i = \bar{\varphi}_i$). The combination of the last equation with Eq. (2.10) produces Eq. (2.1), as expected. To obtain Eq. (2.5), several additional steps are needed. First, one introduces the functional $\Gamma[\{m_i\}]$ via the Legendre transform

$$\Gamma[\{m_i\}] = S[\bar{\varphi}_i, h_i] + \sum_i h_i(m_j) m_i \tag{2.12}$$

so that the equation of state can be written as

$$h_i = \frac{\delta \Gamma}{\delta m_i} \tag{2.13}$$

The explicit form of Γ at the mean field saddle point level is easily obtained with the help of Eqs. (2.7)–(2.11) and Eq. (2.1) presented in the form

$$h_i = -\sum_j J_{ij} m_j + \frac{1}{\beta} \tanh^{-1} m_i \tag{2.14}$$

Combined use of Eqs. (2.8)–(2.11) and (2.14) produces a result, Eq. (2.5), without the last term. The last term can be obtained if the fluctuations around the mean field $\bar{\varphi}_i$ are taken into account. The use of Eq. (2.7) produces

$$\begin{aligned} \left.\frac{\delta^2 S}{\delta\varphi_i\, \delta\varphi_j}\right|_{\varphi_i = \bar{\varphi}_i} &= J_{ij}^{-1} - \beta(1 - \tanh^2 \beta\bar{\varphi}_i)\, \delta_{ij} \\ &= J_{ij}^{-1} - \beta(1 - m_i^2)\, \delta_{ij} \end{aligned} \tag{2.15}$$

where, in going from the first line to the second, Eqs. (2.9) and (2.11) were used. With the help of Eq. (2.15) one can easily obtain instead of Eq. (2.5) the following result[14]:

$$\begin{aligned} \Gamma[\{m_i\}] &\equiv \mathscr{F}[\{m_i\}] \\ &= \mathscr{F}_{MF}[\{m_i\}] + \frac{1}{2\beta} \ln \det[\delta_{ij} - \beta(1 - m_i^2) J_{ij}] \end{aligned} \tag{2.16}$$

where $\mathscr{F}_{MF}$ denotes all terms, except the last, in Eq. (2.5). Using the fact that for any nonsingular matrix A, $\ln \det \mathrm{A} = \operatorname{tr} \ln \mathrm{A}$, and $\ln(1-x) = -(x + x^2/2 + \cdots)$, one expands the last term in Eq. (2.16). Keeping the terms through quadratic in J_{ij}, and taking into account that $J_{ii} = 0$, one obtains back Eq. (2.5) and, whence, the result (2.4) [or (2.3)]. The above derivation is purely formal and is not based on any plausible physical arguments. This turns out to be crucial in extending the above results to the case of the Potts model.

3. PATH INTEGRAL REPRESENTATION FOR THE POTTS MODEL AND THE MEAN FIELD APPROXIMATION

The tradtional path integral representation of the Potts model is given in the paper by Zia and Wallace.[2] Their formulation is not suitable for obtaining the mean field results (for values of the order parameter outside the critical region) and the Onsager reaction field corrections. Here I provide another path integral treatment of the Potts model which directly generalizes the results for the Ising model presented in Section 2. The partition function for the q-state Potts model can be defined as[16,17]

$$Z = \operatorname*{Tr}_{\{\lambda\}} \exp\left[\frac{1}{2} \sum_{i,j}' K_{ij}\, \delta_{\lambda_i, \lambda_j} + \sum_i h_i(q\, \delta_{\lambda_i, 1} - 1)/(q-1)\right] \tag{3.1}$$

Here the Potts variable λ takes the values $\lambda = 1, \omega, \ldots, \omega^{q-1}$, $\omega = \exp(2\pi i/q)$. Typically K_{ij} is nonzero only for the nearest neighbors. The coupling constant K_{ij} and the magnetic field h_i are explicitly dimensionless and $\{\lambda\}$ denotes, as usual, summation over the Potts variables λ at each lattice site. To develop the path integral formalism two important identities are used[17]

$$\delta_{\lambda_i, \lambda_j} = \frac{1}{q} \sum_{r=0}^{q-1} \lambda_i^r \lambda_j^{q-r} \tag{3.2}$$

$$\delta_{\lambda_i, 1} = \frac{1}{q} \sum_{r=0}^{q-1} \lambda_i^r \tag{3.3}$$

Noticing that $\lambda^q = 1$, one can rewrite the partition function (3.1) as follows:

$$Z = \operatorname*{Tr}_{\{\lambda\}} \exp\left(\frac{1}{2q} \sum_{i,i}' K_{ij} \sum_{r=0}^{q-1} \lambda_i^r \lambda_j^{-r} + \sum_i \frac{h_i}{q-1} \sum_{r=1}^{q-1} \lambda_i^r\right) \tag{3.4}$$

Taking into account that $\lambda^{-r} = (\lambda^*)^r$ and using the Hubbard–Stratonovich identity (with, perhaps, regularized K_{ij}),[15] one arrives at the following result:

$$Z = \operatorname*{Tr}_{\{\lambda\}} \mathcal{N} \int D[\boldsymbol{\varphi}, \boldsymbol{\varphi}^*] \exp\left[-\sum_{i,j} \sum_{r=0}^{q-1} \varphi_i^{*(r)} K_{ij}^{-1} \varphi_j^{(r)} + \sum_i \sum_{r=0}^{q-1} \frac{1}{(2q)^{1/2}} (\varphi_i^{*(r)} \lambda_i^r + \varphi_i^{(r)} \lambda_i^{*r}) + \sum_i \frac{h_i}{q-1} \sum_{r=1}^{q-1} \lambda_i^r \right] \tag{3.5}$$

where $\boldsymbol{\varphi} = \{\varphi^{(1)}, \ldots, \varphi^{(q-1)}\}$, with the corresponding result for the complex conjugate. In the subsequent work it is convenient to rescale the fields $\varphi_i \to \varphi_i/(2q)^{1/2}$. In order to obtain the mean field results, the following ansatz should be used: $\varphi_i^{(0)} \to \varphi_i^{(0)}$, $\varphi_i^{(r)} \to \varphi_i^{(1)}$ for $1 \leqslant r \leqslant q-1$. For the fixed i consider now the following average:

$$F = \operatorname*{Tr}_{\{\lambda\}} \exp\left[\frac{1}{2q} (\varphi^{(0)} + \varphi^{*(0)}) + \frac{\varphi^{*(1)}}{2q} \sum_{r=1}^{q-1} \lambda^r + \frac{\varphi^{(1)}}{2q} \sum_{r=1}^{q-1} \lambda^{-r} + \frac{h}{q-1} \sum_{r=1}^{q-1} \lambda^r \right] \tag{3.6}$$

where the i dependence of the fields is temporarily suppressed. Evidently, the exponential term which contains the $\varphi^{(0)}$ ($\varphi^{*(0)}$) field can be taken outside the trace. For this field the Hubbard–Stratonovich transformation can be performed backward, resulting in some unimportant constant which I shall ignore. To perform the summation over the set $\{\lambda\}$ in Eq. (3.6), the following equation is of great importance:

$$\sum_{r=1}^{q-1} \lambda^r = \frac{\lambda^{q-1}\lambda - \lambda}{\lambda - 1} = \delta_{\lambda,1}(q-1) - (1 - \delta_{\lambda,1}) \tag{3.7}$$

With the help of Eq. (3.7), the summation over λ in Eq. (3.6) can be readily performed, thus producing the final result,

$$F = \exp\left[\left(\frac{q-1}{2q} \right) (\varphi^{(1)} + \varphi^{*(1)}) + h \right] + (q-1) \exp\left[-\frac{h}{q-1} - \frac{1}{2q} (\varphi^{(1)} + \varphi^{*(1)}) \right] \tag{3.8}$$

The mean field partition function for the Potts model can now be represented as

$$Z_{MF} = \mathcal{N} \int D[\varphi^{(1)}, \varphi^{*(1)}] \exp\{ -S[\varphi^{(1)}, \varphi^{*(1)}] \} \tag{3.9}$$

where the ation S is given by

$$S[\varphi^{(1)}, \varphi^{*(1)}] = \frac{q-1}{2q} \sum_{i,j} \varphi^{*(1)} K_{ij}^{-1} \varphi_j$$

$$- \sum_i \ln \left\{ \exp\left[\frac{q-1}{2q} (\varphi_i^{(1)} + \varphi_i^{*(1)}) + h \right] \right.$$

$$\left. + (q-1) \exp\left(-\frac{h_i}{q-1} - \frac{\varphi_i^{(1)} + \varphi_i^{*(1)}}{2q} \right) \right\} \quad (3.10)$$

Minimization of the action produces

$$\sum_j K_{ij}^{-1} \bar{\varphi}_j^{(1)} = \frac{y_{1i} - y_{2i}}{y_{1i} + (q-1) y_{2i}} \quad (3.11)$$

with a similar expression for the complex conjugate field. Here

$$y_{1i} = \exp\left[\frac{q-1}{2q} (\bar{\varphi}_i^{(1)} + \bar{\varphi}_i^{*(1)}) + h_i \right]$$

$$y_{2i} = \exp\left[-\frac{h_i}{q-1} - \frac{1}{2q} (\bar{\varphi}_i^{(1)} + \bar{\varphi}_i^{*(1)}) \right] \quad (3.12)$$

and the overbars denote the mean field saddle point approximation as in Section 2. On the other hand, as in the case of the Ising model, the magnetization m_i at the mean field level is given by

$$m_i = \frac{\partial \ln \hat{Z}_{MF}}{\partial h_i} = \frac{y_{1i} - y_{2i}}{y_{1i} + (q-1) y_{2i}} \quad (3.13)$$

where $\hat{Z}_{MF} \simeq \exp\{-S[\bar{\varphi}^{(1)}, \bar{\varphi}^{*(1)}]\}$.

Combining Eqs. (3.11) and (3.13) produces

$$\varphi_i^{-(1)} = \sum_j K_{ij} m_j \quad (3.14)$$

with a similar expression for $\bar{\varphi}_i^{*(1)}$. Using Eqs. (3.12)–(3.14), one easily obtains

$$m_i = \frac{\hat{y}_{1i} - \hat{y}_{2i}}{\hat{y}_{1i} + (q-1) \hat{y}_{2i}} \quad (3.15)$$

where

$$\hat{y}_{1i} = \exp\left(\sum_j K_{ij} m_j + h_i \right) \quad (3.16a)$$

$$\hat{y}_{2i} = \exp\left(-\frac{h_i}{q-1} \right) \quad (3.16b)$$

This can be somewhat rearranged to produce

$$h_i = \frac{q-1}{q} \ln\left(\frac{1+(q-1)m_i}{1-m_i}\right) - \frac{q-1}{q} \sum_j K_{ij} m_j \tag{3.17}$$

The last result coincides with that given in ref. 16. For $q=2$ (the Ising model case) one obtains back Eq. (2.4) (without the reaction field term) if one recalls that $\tanh^{-1} x = 1/2 \ln [(1+x)/(1-x)]$.

As before [see Eq. (2.12)], it is useful to introduce the thermodynamic potential $\Gamma[\{m_i\}]$ via a Legendre transform. Using Eqs. (3.10), (3.14), and (3.17), one obtains after some algebra, instead of Eq. (2.12), the following result:

$$\begin{aligned} \beta\Gamma[\{m_i\}] &\equiv \beta\mathscr{F}[\{m_i\}] \\ &= -\frac{1}{2}\frac{q-1}{q} \sum_{i,j}{}' m_i K_{ij} m_j + \frac{1}{q} \sum_i \{[1+(q-1)m_i] \\ &\quad \times \ln[1+(q-1)m_i] + (q-1)(1-m_i)\ln(1-m_i)\} \end{aligned} \tag{3.18}$$

where the unimportant constant $-N \ln q$ was omitted. The last result coincides with that given in refs. 9 and 16. The Ising model case $(q=2)$ is obtained in conventional notations[14] if one redefines the coupling constant $K_{ij}/2 \to K_{ij}$, or, for general q, $K_{ij}/q \to K_{ij}$. The given analysis accomplishes the mean field saddle point treatment of the Potts model.

4. ONSAGER REACTION FIELD FOR THE POTTS MODEL

In Section 2 the use of the path integral methods made it possible to obtain in a systematic fashion the one-loop fluctuation corrections to the mean field results for the Ising model. In developing the above methods for the Potts model, it is essential to reproduce the Ising model results $(q=2)$. This is essential for two reasons: first, for requirements of consistency, and, second, because the previous treatments[8] ignored the consistency requirement, which resulted in rather cumbersome and inconclusive results.

Taking into account Eq. (2.15), in the present case one has to calculate the fluctuation matrix obtained from the following functional:

$$\begin{aligned} S[\boldsymbol{\varphi}, \boldsymbol{\varphi}^*] &= \sum_{i,j} \frac{1}{2q} \sum_{r=1}^{q-1} \varphi_i^{*(r)} K_{ij}^{-1} \varphi_j^{(r)} \\ &\quad - \sum_i \ln \operatorname*{Tr}_{\{\lambda\}} \exp\left\{\frac{1}{2q} \sum_{r=1}^{q-1} \left[\varphi_i^{*(r)} \lambda_i^r + \varphi_i^{(r)} \lambda_i^{*r} + \frac{h_i}{q-1} \lambda_i^r\right]\right\} \end{aligned} \tag{4.1}$$

To develop the general method of calculation, it is instructive to consider first the Ising model case, $q=2$. In this case the Tr in Eq. (4.1) is easily performed and one obtains, after obvious rescaling of the fields $\varphi_i \leftrightarrow \varphi_i/2$ and the coupling constant K_{ij}, the following familiar result $(h_i=0)$[14]:

$$S_{\text{Ising}} = \frac{1}{2}\sum_{i,j} \varphi_{1i} K_{ij}^{-1} \varphi_{1j} + \frac{1}{2}\sum_{i,j} \varphi_{2i} K_{ij}^{-1} \varphi_{2j} - \sum_i \ln(2\cosh\varphi_{1i}) \tag{4.2}$$

In arriving at Eq. (4.2), the field $\varphi_i^{(1)}$ was represented as $\varphi_i^{(1)} = \varphi_{1i} + i\varphi_{2i}$, etc. Minimization of Eq. (4.2) produces back Eq. (2.15), this time written in the form

$$\left.\frac{\delta^2 S_{\text{Ising}}}{\delta\varphi_{1i}\,\delta\varphi_{1j}}\right|_{\varphi_{1i}=\bar{\varphi}_i} = K_{ij}^{-1} - (1-m_i^2)\delta_{ij} \tag{4.3}$$

Use of Eq. (3.11) indicates that

$$\left.\frac{\delta^2 S_{\text{Ising}}}{\delta\varphi_{1i}\,\delta\varphi_{2j}}\right|_{\substack{\varphi_{1i}=\bar{\varphi}_{1i}\\ \varphi_{2i}=\bar{\varphi}_{2i}}} = 0 \tag{4.4}$$

while Eq. (4.2) yields

$$\left.\frac{\delta^2 S_{\text{Ising}}}{\delta\varphi_{2i}\,\delta\varphi_{2j}}\right|_{\varphi_{2i}=\bar{\varphi}_{2i}} = K_{ij}^{-1} \tag{4.5}$$

The last fluctuation matrix is of no importance, however, because it does not contain the order parameter.

Going back to the general case of arbitrary q, rescaling the fields in Eq. (4.1), and using the real fields instead of the complex, one can rewrite Eq. (4.1) in the following form (note: $K_{ij} \leftrightarrow K_{ij}/q$, $h=0$, $\varphi \leftrightarrow \varphi/q$):

$$\begin{aligned} S[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] = {} & \frac{1}{2}\sum_{i,j}\sum_{r=1}^{q-1} \varphi_{1i}^{(r)} K_{ij}^{-1} \varphi_{1j}^{(r)} + \frac{1}{2}\sum_{i,j}\sum_{r=1}^{q-1} \varphi_{2i}^{(r)} K_{ij}^{-1} \varphi_{2j}^{(r)} \\ & - \sum_i \ln \operatorname*{Tr}_{\{\lambda\}} \exp\left\{\frac{1}{2}\sum_{r=1}^{q-1} [\varphi_{1i}^{(r)}(\lambda_i^r + \lambda_i^{-r}) + i\varphi_{2i}^{(r)}(\lambda_i^{-r} - \lambda_i^r)]\right\} \end{aligned} \tag{4.6}$$

Now one has to calculate the following matrix elements:

$$\left.\frac{\delta^2 S}{\delta\varphi_{1i}^{(r)}\,\delta\varphi_{1j}^{(r)}}\right|_{\varphi_{1i}^{(r)}=\bar{\varphi}_{1i}}, \quad \left.\frac{\delta^2 S}{\delta\varphi_{1i}^{(r)}\,\delta\varphi_{1j}^{(l)}}\right|_{\varphi_{1i}^{(r,l)}=\bar{\varphi}_{1i}}, \ldots \tag{4.7}$$

This can be accomplished if one recognizes that, while taking the functional derivatives using functional (4.6), only the fields with the *fixed* "color" index r (or r and l) should not be of the mean field type from the outset. To calculate the second functional derivative, which, for example, is diagonal in color index r, consider the following auxiliary functional (the spatial index i is suppressed):

$$I = \operatorname*{Tr}_{\{\lambda\}} \exp\{\bar{\varphi}_1[\delta_{\lambda,1}(q-1) - (1-\delta_{\lambda,1})] + (\varphi_1^{(r)} - \bar{\varphi}_1)\tfrac{1}{2}(\lambda^r + \lambda^{-r})\} \tag{4.8}$$

Here Eqs. (3.6) and (3.7) were used. The trace now can be easily computed, thus yielding the final result

$$\begin{aligned} I = &\exp[(q-1)\bar{\varphi}_1 + (\varphi_1^{(r)} - \bar{\varphi}_1)] \\ &+ \exp\{-\bar{\varphi}_1\}\left\{\sum_{i=1}^{q-1} \exp\left[\frac{1}{2}(\varphi_1^{(r)} - \bar{\varphi}_1)(\omega^{ir} + \omega^{-ir})\right]\right\} \end{aligned} \tag{4.9}$$

Taking the logarithm of I and functionally differentiating, one obtains,

$$\left.\frac{\delta}{\delta\varphi_i^{(r)}} \ln I\right|_{\varphi_1^{(r)} = \bar{\varphi}_1} = m \tag{4.10}$$

where Eqs. (3.12)–(3.16) were used along with an obvious result

$$\sum_{i=1}^{q-1} \omega^{ir} = -1 \tag{4.11}$$

Before taking the second functional derivative, however, it is useful to remember that

$$\sum_{i=1}^{q-1} \omega^{2ri} = \begin{cases} 1, & q=2 \\ -1, & q \neq 2 \end{cases} \tag{4.12}$$

which comes directly from the definition of ω and the fact that for $q=2$ the index r can be only one. Using Eqs. (4.13), (4.10), and (4.12), one easily obtains

$$\begin{aligned} \left.\frac{\delta^2 S}{\delta\varphi_i^{(r)}\,\delta\varphi_j^{(r)}}\right|_{\varphi_i^{(r)} = \bar{\varphi}_i} = &K_{ij}^{-1} - \mathcal{N}^{-1}\{e^{a_1} + e^{a_2}(\tfrac{1}{2}) \\ &\times [(2\delta_{q,2} - 1) + q - 1]\}\,\delta_{ij} + m_i^2\delta_{ij} \end{aligned} \tag{4.13}$$

where

$$\mathcal{N} = e^{a_1} + (q-1)e^{a_2}, \quad e^{a_1} = \exp[(q-1)\bar{\varphi}_1], \quad e^{a_2} = \exp(-\bar{\varphi}_1)$$

and $\bar{\varphi}_1$ is given by Eq. (3.14) while $\bar{\varphi}_2 = 0$ at the mean field level. For $q = 2$ the old result, Eq. (4.3), is readily reproduced, as expected. Computation of other functional derivatives proceeds analogously. In particular, one has $(r \neq 1)$

$$\left.\frac{\delta^2 S}{\delta\varphi_i^{(r)}\,\delta\varphi_{1j}^{(l)}}\right|_{\varphi_{1i}^{(r,l)} = \bar{\varphi}_{1i}} = -\mathcal{N}^{-1} m_i(1-m_i)\,\delta_{ij} \tag{4.14}$$

also

$$\left.\frac{\delta^2 S}{\delta\varphi_{2i}^{(r)}\,\delta\varphi_{2j}^{(r)}}\right|_{\varphi_{2i}^{(r)} = \bar{\varphi}^{(2)} = 0} = K_{ij}^{-1} + \mathcal{N}^{-1}\frac{e^{a_2}}{2}\left[(2\delta_{q,2}-1)-(q-1)\right] \tag{4.15a}$$

and

$$\left.\frac{\delta^2 S}{\delta\varphi_{2i}^{(r)}\,\varphi_{2j}^{(l)}}\right|_{\varphi_{2i}^{(r)} = \bar{\varphi}^{(2)} = 0} = 0 \tag{4.15b}$$

Finally, for the cross terms, one obtains

$$\left.\frac{\delta^2 S}{\delta\varphi_{1i}^{(r)}\,\delta\varphi_{2j}^{(l)}}\right|_{\substack{\varphi_{1i}^{(r)} = \bar{\varphi}_i \\ \varphi_{2i}^{(r)} = \bar{\varphi}_i^{(2)} = 0}} = 0 \tag{4.16}$$

for all r and l.

Equation (4.16) indicates that the fluctuation matrix has the block-diagonal form, which significantly simplifies the rest of the calculations. In particular, I begin with a computation of the one-loop correction to the mean field free energy result, Eq. (3.18). This correction is the most easily computed for the case of uniform magnetization $m_i = m$. In this case, the use of Fourier transform methods[1] by analogy with the Ising model case produces the following results for the corresponding fluctuation matrices (for the fixed wave vector $\mathbf{k}$)

$$\left\|\frac{\delta^2 S}{\delta\boldsymbol{\varphi}_1^{(r)}\,\delta\boldsymbol{\varphi}_1^{(l)}}\right\| = \begin{array}{c} \begin{matrix} 1 & 2 & \cdots & q-1 \end{matrix} \\ \begin{pmatrix} A_1 & A_2 & \cdots & A_2 \\ A_2 & A_1 & \cdots & A_2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & A_1 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ q-1 \end{array} \tag{4.17}$$

where

$$A_1 = [K(\mathbf{k})]^{-1} - \mathcal{N}^{-1}\{e^{a_1} + e^{a_2}(1/2)[(2\delta_{q,2}-1)+q-1]\} + m^2$$

and $a_1 = (q-1)\,zKm$, $a_2 = -zKm$, where z is the coordination number of the lattice and K is the strength of the nearest neighbor bond interaction; $A_2 = -\mathcal{N}^{-1}m(1-m)$. Also, one has

$$\left\| \frac{\delta^2 S}{\delta\varphi_2^{(r)}\,\delta\varphi_2^{(l)}} \right\| = \begin{array}{cc} \begin{array}{cccc} 1 & 2 & \cdots & q-1 \end{array} & \\ \begin{pmatrix} A_3 & 0 & \cdots & 0 \\ 0 & A_3 & \cdots & \cdot \\ \cdot & \cdot & \ddots & \cdot \\ 0 & \cdot & \cdots & A_3 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ \vdots \\ q-1 \end{array} \end{array} \tag{4.18}$$

where

$$A_3 = [K(\mathbf{k})]^{-1} + \mathcal{N}^{-1}e^{a_2}(1/2)[(2\delta_{q,2}-1)-(q-1)] \equiv \hat{\lambda}(\mathbf{k}) \tag{4.19}$$

As usual, one subsequently expands $K^{-1}(\mathbf{k}) \approx K^{-1}(0) + \alpha k^2$, where α is related to the coordination number of the lattice.[1,14,15] The determinant of the matrix given by Eq. (4.17) is a circulant and can be easily diagonalized,[18] with the following results for eigenvalues:

$$\lambda_1(\mathbf{k}) = A_1 + (q-2)A_2 \tag{4.20}$$

$$\lambda_i(\mathbf{k}) = A_1 - A_2 \tag{4.21}$$

for $i = 2, 3,..., q-1$.

The matrix given by Eq. (4.18) is already diagonal. The use of Eqs. (4.18)–(4.21) allows one to write for the fluctuation corrections to the free energy the following result:

$$\beta\mathcal{F}_{fl} = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d} \ln \lambda_1(\mathbf{k}) + \frac{q-2}{2}\int \frac{d^dk}{(2\pi)^d} \ln \lambda_2(\mathbf{k}) + \frac{q-1}{2}\int \frac{d^dk}{(2\pi)^d} \ln \hat{\lambda}(\mathbf{k}) \tag{4.22}$$

This result should be added to the homogeneous form of Eq. (3.18). Equation (4.22) produces automatically the Ising model result for $q = 2$[1,14] and is valid for arbitrary q. One can study, in principle, at this point how the one-loop fluctuation corrections affect the value of q at which the first-order phase transition takes place.[8] Instead, I provide below the derivation of the Onsager reaction field terms. To this purpose, it is useful *not* do diagonalize the fluctuation matrices in **k**-space, which allows one to obtain a closed-form result, Eq. (4.22), for the fluctuation corrections. Using results of Section 3, it is necessary to consider the off-diagonal terms, Eq. (4.14), perturbatively. Averaging with the Gaussian measure given by the kernel (4.13) allows one to easily obtain the lowest order off-diagonal fluctuation correction to the free energy given by

$$\beta \mathscr{F}_{\mathrm{fl}}^{\mathrm{od}} = (q-1)(q-2) \sum_{i,j} \hat{K}_{ij} m_j (1-m_j)\, \hat{K}_{ji} (1-m_i) m_i \tag{4.23}$$

where

$$\begin{aligned} \hat{K}_{ij} &= [K^{-1} - \mathscr{N}^{-1} \{ e^{a_1} + e^{a_2} \\ &\quad \times \tfrac{1}{2} [(2\delta_{q,2} - 1) + q - 1] I + \mathbf{m}^2 I]_{ij}^{-1} \\ &= \sum_l K_{il} [I - \mathscr{N}^{-1} \{ e^{a_1} + e^{a_2} \\ &\quad \times \tfrac{1}{2} [(2\delta_{q,2} - 1) + q - 1] K + \mathbf{m}^2 K]_{lj}^{-1} \\ &\approx K_{ij} + K_{ij}^2 \mathscr{N}^{-1} \{ e^{a_1} + e^{a_2} \\ &\quad \times \tfrac{1}{2} [(2\delta_{q,2} - 1) + q - 1] \} - K_{ij}^2 m_i + \cdots \end{aligned} \tag{4.24}$$

In view of Eqs. (4.23) and (4.24) it is sufficient to write

$$\hat{K}_{ij} \approx K_{ij} \tag{4.25}$$

The higher order terms could be included, if necessary. The use of Eq. (4.25) brings the final expression for the off-diagonal contribution to the Onsager's reaction field terms,

$$\beta \mathscr{F}_{\mathrm{fl}}^{\mathrm{od}} = (q-1)(q-2) \sum_{i,j} K_{ij} m_j (1-m_j)\, K_{ji} (1-m_i) m_i \tag{4.26}$$

Consider now the diagonal contributions. Using Eq. (4.13) along with formula $\ln \det \mathsf{A} = \operatorname{tr} \ln \mathsf{A}$ and expanding the logarithm, one obtains [with accuracy up to $O(K_{ij}^2)$ terms], by analogy with the Ising model case (see Section 3), the following result:

$$\begin{aligned} \beta \mathscr{F}_{\mathrm{fl}}^{d_1} &= \frac{q-1}{2} \operatorname{tr} \ln \left\{ (\delta_{ij} - \frac{1}{q} \left(1 + \frac{q}{2} \right) + \left(\frac{q}{2} - \delta_{q,2} \right) \sum_f K_{if} m_f \right] K_{ij} + m_i^2 K_{ij} \right\} \\ &\approx -\frac{q-1}{4q^2} \sum_{i,j}{}' \left(1 + \frac{q}{2} - q m_i^2 \right) K_{ij}^2 \left(1 + \frac{q}{2} - q m_j^2 \right) + O(K_{ij}^3) \end{aligned} \tag{4.27}$$

In arriving at Eq. (4.27), the exponential factors in Eq. (4.13) were expanded and only the lowest orders in K_{ij} terms were kept. Consider finally the contribution coming from the fluctuation kernel, Eq. (4.15). Using the same kind of approximations as in the previous case, one obtains

$$\beta \mathscr{F}_{\mathrm{fl}}^{d_2} = \frac{q-1}{2} \operatorname{tr} \ln \left[\delta_{ij} + \left(\delta_{q,2} - \frac{q}{2} \right) \frac{1}{q} \left(1 - \sum_l K_{il} m_l \right) K_{ij} \right] \tag{4.28}$$

This term produces, however, zero if the accuracy up to $O(K_{ij})$ is desired (because $K_{ii}=0$ by definition). Collecting all terms [see Eqs. (3.18), (4.26)–(4.28)] one obtains the free energy with Onsager's corrections in the following form:

$$\begin{aligned}\beta\mathscr{F} = & -\frac{1}{2}\frac{q-1}{q}\sum_{i,j}' m_i K_{ij} m_j + \frac{1}{q}\sum_i \{[1+(q-1)m_i] \\ & \times \ln[1+(q-1)m_i] + (q-1)(1-m_i)\ln(1-m_i)\} \\ & -\frac{q-1}{4q^2}\sum_{i,j}'\left(1+\frac{q}{2}-qm_i^2\right)\frac{K_{ij}^2}{q^2}\left(1+\frac{q}{2}-qm_j^2\right) \\ & +(q-1)(q-2)\sum_{i,j}' K_{ij} m_j(1-m_j) \\ & \times K_{ji}(1-m_i)m_i + O(K_{ij}^3) \end{aligned} \tag{4.29}$$

Here the rescaling of the fields φ and the coupling constant K defined before Eq. (4.6) was taken into account. Notice that the inclusion of these Onsager terms does not change the mean field prediction about the first-order phase transition for $q>2$ coming from the mean field analysis,[9] so that the exact result $q_c=4$ for $d=2$[10] should be understood, as before, in the asymptotic sense of large-q expansions.[9] Equation (4.29) coincides with Eq. (2.5) for $q=2$, as expected (if the term $K_{ij}/2 \to K_{ij}$, as stated in Section 3). In case the problem of percolation ($q=1$) is considered, the above expression for the free energy should actually be divided by the factor of $q-1$. For more details, see refs. 9 and 19.

5. DISCUSSION

In this paper a new systematic path integral treatment of the Potts model has been developed. This treatment can be considered as a direct extension of the similar treatment for the Ising model.[14] Previous path integral treatments of the Potts model are based on the work by Zia and Wallace.[2] Their method is not suitable, however, for obtaining the systematic mean field corrections in cases when the order parameter is not small, so that the functional integral treatment of the Potts model was noticeably different from that of the Ising model. I hope that the above derivation of the Onsager reaction field corrections for the Potts model will be especially useful in those cases when the randomness is important. In particular, using the result of the previous section, it is now rather straightforward to develop the TAP approach[12] for the Potts spin glass. The results are especially useful in determining the correct Potts spin-glass transition temperature.[5,12] It is well known that the inclusion of the

Onsager reaction field corrections within the TAP approach to the mean field Ising model produces results which are in agreement with that obtained by Sherington and Kirkpatrick[11] by replica-trick methods. The various real-space renormalization group methods[19] might benefit also from the presented development.

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REFERENCES

1. D. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena* (McGraw-Hill, New York, 1978).
2. R. Zia and D. Wallace, *J. Phys. A* **8**:1495 (1975).
3. D. Wallace and A. Young, *Phys. Rev. B* **17**:2384 (1978).
4. M. Barbosa, M. Gusmao, and W. Treuman, *Phys. Rev. B* **34**:3165 (1986).
5. K. Binder and A. Young, *Rev. Mod. Phys.* **58**:801 (1986).
6. D. Thouless, P. Anderson, and R. Palmer, *Phil. Mag.* **35**:593 (1971).
7. D. Thirumalai and T. Kirkpatrick, *Phys. Rev. B* **38**:4881 (1988).
8. R. Dekeyser, A. Maritan, and A. Stella, *J. Phys. A* **16**:1713 (1983).
9. F. Wu, *Rev. Mod. Phys.* **54**:235 (1982).
10. R. Baxter, *J. Phys. C* **6**:L445 (1973).
11. D. Sherington and S. Kirkpatrick, *Phys. Rev. Lett.* **35**:1972 (1975).
12. M. Mezard, G. Parisi, and M. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
13. L. Onsager, *J. Am. Chem. Soc.* **58**:1486 (1936).
14. J. Negele and H. Orland, *Quantum Many Particle Systems* (Addision-Wesley, New York, 1988).
15. M. Fisher, in *Lecture Notes in Physics*, Vol. 186 (Springer, Berling, 1983).
16. C. Dasgupta and R. Pandit, *J. Stat. Phys.* **47**:375 (1987).
17. L. Mittag and M. Stephen, *J. Phys. A* **7**:L109 (1976).
18. C. Tompson, *Classical Equilibrium Statistical Mechanics* (Clarendon Press, Oxford, 1988).
19. T. Lubensky, in *Les Houches, XXXI*, (North-Holland, Amsterdam, 1979), pp.408–472.
20. A. Kholodenko, *J. Phys. A* **18**:3227 (1985).